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**APPROXIMATIONS OF PERIODIC FUNCTIONS BY
ANALOGUE OF ZIGMUND'S SUMS IN THE SPACES $L^{p(\cdot)}$**

In this work we found order estimates for the upper bounds of the deviations of analogue of Zigmund's sums on the classes of $(\psi; \beta)$ -differentiable functions in the metrics of generalized Lebesgue spaces with variable exponent.

1. Definition and formulation of the problem. Let $p = p(x)$ be a 2π -periodic measurable and essentially bounded function and let $L^{p(\cdot)}$ be space of measurable 2π -periodic functions f such that

$$\int_{-\pi}^{\pi} |f(x)|^{p(x)} dx < \infty.$$

If $\underline{p} := \operatorname{ess\,inf}_x |p(x)| > 1$ and $\bar{p} := \operatorname{ess\,sup}_x |p(x)| < \infty$, then $L^{p(\cdot)}$ are Banach spaces [1] (see., also [3]) with the norm, which can be given by the formula

$$\|f\|_{p(\cdot)} := \inf \left\{ \alpha > 0 : \int_{-\pi}^{\pi} \left| \frac{f(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}.$$

Here are some definitions which will used in the statement and proof of the results of this article.

Definition 1. *It is said that a function $p = p(x)$ satisfies the Dini-Lipschitz condition of order γ , if*

$$\omega(p; \delta) \left(\ln \frac{1}{\delta} \right)^\gamma \leq K, \quad 0 < \delta < 1,$$

where

$$\omega(p; \delta) = \sup_{x_1, x_2 \in [-\pi; \pi]} \left\{ |p(x_1) - p(x_2)| : |x_1 - x_2| \leq \delta \right\}.$$

The set of 2π -periodic exponents $p = p(x) > 1$, satisfying the Dini-Lipschitz condition of order $\gamma \geq 1$ in the period, is denoted by \mathcal{P}^γ . Obviously, if $p \in \mathcal{P}^\gamma$, then $\underline{p} > 1$ and $\bar{p} < \infty$.

In the work [1] shown that when $1 < \underline{p}$, $\bar{p} < \infty$, space $L^{q(\cdot)}$, where $q(x) = \frac{p(x)}{p(x)-1}$, is conjugate for $L^{p(\cdot)}$ and for arbitrary functions $f \in L^{p(\cdot)}$ and $g \in L^{q(\cdot)}$ an analogue of the classical Hölder's inequality is true:

$$\int_{-\pi}^{\pi} |f(x)g(x)| dx \leq K_{p,q} \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}, \quad (K_{p,q} \leq 1/\underline{p} + 1/\underline{q}), \quad (1)$$

which, in particular, implies embedding: $L^{p(\cdot)} \subset L$, where L is space of 2π -periodic integrable on the period functions.

The space $L^{p(\cdot)}$ are called generalized Lebesgue spaces with variable exponent. It is clear, that if $p = p(x) = \text{const} > 0$, spaces $L^{p(\cdot)}$ coincide with the classical Lebesgue spaces L_p . In its turn, if $\bar{p} < \infty$, spaces $L^{p(\cdot)}$ are a special case of the so-called space Orlicz-Musielak [4]. For the first time, Lebesgue space with variable exponent appeared in the literature in the article W. Orlicz [5]. In the work [6] spaces $L^{p(\cdot)}$ considered as an example of the more general function spaces and, furthermore, been studied by many authors in different directions. The basic results of the theory of these spaces are available, for example, in [1, 3, 7, 8, 9, 10, 11]. Note also that the generalized Lebesgue spaces with variable exponent used in the theory elastic mechanics, the theory of differential operators, variations calculus [12, 13, 14].

Next, we need the definitions of the $(\psi; \beta)$ -derivative and the sets L_β^ψ , which belongs to A.I. Stepanetz [2, c. 142 – 143].

Definition 2. Let $f \in L$ and

$$S[f] = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f, x) \quad (2)$$

be its Fourier series. Let, further, $\psi(k)$ be arbitrary function of natural argument and $\beta \in \mathbb{R}$. Assume that the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k(f) \cos \left(kx + \frac{\beta\pi}{2} \right) + b_k(f) \sin \left(kx + \frac{\beta\pi}{2} \right) \right) \quad (3)$$

is the Fourier series of some function from L . This function is denoted by $f_\beta^\psi(\cdot)$ (or $(D_\beta^\psi f)(\cdot)$) and called $(\psi; \beta)$ -derivative of a function $f(\cdot)$. The set of functions $f(\cdot)$, satisfying this condition is denoted by L_β^ψ .

Denote by $L_{\beta, p(\cdot)}^\psi$ the classes $(\psi; \beta)$ -differentiable functions $f \in L$, such that $f_\beta^\psi \in L^{p(\cdot)}$, and by $\hat{Z}_n(f; x)$ the trigonometric polynomials of the form

$$\hat{Z}_n(f; x) := \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(1 - \frac{\psi(n)}{\psi(k)} \right) A_k(f; x). \quad (4)$$

In this paper, we study the value

$$\mathcal{E}(L_{\beta, p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} := \sup_{f \in L_{\beta, p(\cdot)}^\psi} \|f - \hat{Z}_n(f)\|_{s(\cdot)}$$

upper bounds of deviations analogues of Zigmund's sums $\hat{Z}_n(f; x)$ on the classes $L_{\beta, p(\cdot)}^\psi := \{f \in L_\beta^\psi L^{p(\cdot)} : f_\beta^\psi \in U_{p(\cdot)}\}$, where $U_{p(\cdot)} := \{\varphi \in L^{p(\cdot)} : \|\varphi\|_{p(\cdot)} \leq 1\}$ — the unit ball of $L^{p(\cdot)}$.

Note that in the case $\psi(k) = 1/k^r$, $r > 0$, the sums (4) are well known Zigmund's sums

$$Z_n(f; x) := \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(1 - \left(\frac{k}{n}\right)^r\right) A_k(f; x).$$

2. Auxiliary results. In the proof of the main assertions of this work we use the following well-known results.

Theorem A. [10] *If $p \in \mathcal{P}^\gamma$, then for an arbitrary function $f \in L^{p(\cdot)}$ the inequalities hold*

$$\|S_n(f)\|_{p(\cdot)} \leq C_p \|f\|_{p(\cdot)}, \quad (5)$$

$$\|\tilde{f}\|_{p(\cdot)} \leq K_p \|f\|_{p(\cdot)}, \quad (6)$$

where $\tilde{f}(\cdot)$ is a functions, trigonometric conjugate to $f(\cdot)$, and C_p and K_p are positive constants which does not depend on n and f .

From the inequality (5) of Theorem A, in particular, follows that for an arbitrary function $f \in L^{p(\cdot)}$, on condition $p \in \mathcal{P}^\gamma$, its Fourier series converges to f in the metric of the spaces $L^{p(\cdot)}$, that is

$$\|f - S_n(f)\|_{p(\cdot)} \rightarrow 0, \quad n \rightarrow \infty, \quad (7)$$

and the relation holds

$$E_n(f)_{p(\cdot)} \leq \|f - S_{n-1}(f)\|_{p(\cdot)} \leq K_p E_n(f)_{p(\cdot)}, \quad (8)$$

where

$$E_n(\varphi)_{p(\cdot)} := \inf_{t_{n-1} \in \mathcal{T}_{n-1}} \|\varphi - t_{n-1}\|_{p(\cdot)}, \quad \varphi \in L^{p(\cdot)},$$

is the best approximation of φ by subspace \mathcal{T}_{n-1} trigonometric polynomials of order, not higher than $n - 1$, and K_p is value which depends only on $p = p(x)$.

Lemma A. [15] *Let the sequence $\mu(k)$, $k = 0, 1, 2, \dots$, satisfies the conditions*

$$\nu_0 = \nu_0(\mu) = \sup_k |\mu(k)| \leq C, \quad \sigma_0 = \sigma_0(\mu) = \sup_{m \in \mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\mu(k+1) - \mu(k)| \leq C,$$

where C is value which does not depend on k and m .

Then, if $p \in \mathcal{P}^\gamma$, for a given function $f \in L^{p(\cdot)}$ there exists a function $F \in L^{p(\cdot)}$ such that the series

$$\frac{\mu(0)a_0(f)}{2} + \sum_{k=1}^{\infty} \mu(k)(a_k(f) \cos kx + b_k(f) \sin kx)$$

is the Fourier series of F and the inequality is true

$$\|F\|_{p(\cdot)} \leq K\lambda\|f\|_{p(\cdot)}, \quad \lambda = \max\{\nu_0, \sigma_0\}, \quad (9)$$

where the value K does not depend on the function f .

In the case $p = p(x) \equiv \text{const}$ this statement is a well-known lemma of Marcinkiewicz for multipliers [16].

We will also use the following theorem of Hardy-Littlewood.

Theorem B. [17] Let $1 < p < s < \infty$, $p, s = \text{const}$, $\alpha = p^{-1} - s^{-1}$ and

$$D_\alpha(t) = \sum_{k=1}^{\infty} k^{-\alpha} \cos kt.$$

Then, for an arbitrary function $\varphi \in L_p$ the convolution

$$\Phi_\alpha(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x+t) D_\alpha(t) dt$$

belongs to L_s , and

$$\|\Phi_\alpha\|_s \leq C_{s,p} \|\varphi\|_p,$$

where the value $C_{s,p}$ depends only on s and p .

Note that if $\varphi \in L_p$ and $S[\varphi] = \sum_{k=0}^{\infty} A_k(\varphi; x)$, then

$$S[\Phi_\alpha] = \sum_{k=0}^{\infty} k^{-\alpha} A_k(\varphi; x),$$

that is $\Phi_\alpha = M_\alpha(\varphi)$, where M_α — operator-multiplier, which is determined by the sequence of $\mu_\alpha(k) = k^{-\alpha}$, $k = 0, 1, 2, \dots$, and it acts from L_p to L_s , where indicators $1 < p < s < \infty$, $p, s = \text{const}$, are related by the equation $p^{-1} - s^{-1} = \alpha$.

2. Approximation by analogue of Zigmund's sums.

We define the function $\mu_{n,\alpha}(k)$ and $\tilde{\mu}_{n,\alpha}(k)$ as follows:

$$\mu_{n,\alpha}(k) = \begin{cases} k^\alpha \psi(n) \cos \frac{\beta\pi}{2}, & 1 \leq k \leq n-1, \\ k^\alpha \psi(k) \cos \frac{\beta\pi}{2}, & n \leq k; \end{cases} \quad (10)$$

$$\tilde{\mu}_{n,\alpha}(k) = \begin{cases} k^\alpha \psi(n) \sin \frac{\beta\pi}{2}, & 1 \leq k \leq n-1, \\ k^\alpha \psi(k) \sin \frac{\beta\pi}{2}, & n \leq k; \end{cases} \quad (11)$$

For each fixed $\alpha \geq 0$ we denote by $\Upsilon_{\alpha,n}$ the set of pairs $(\psi; \beta)$, such that for any positive number n the conditions

$$\nu_\alpha(\psi; \beta; n) := \sup_k |\mu_{n,\alpha}(k)| \leq C\nu(n)n^\alpha < K, \quad (12)$$

$$\sigma_\alpha(\psi; \beta; n) := \sup_{m \in \mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\mu_{n,\alpha}(k+1) - \mu_{n,\alpha}(k)| \leq C\nu(n)n^\alpha < K, \quad (13)$$

and similar conditions for the function $\tilde{\mu}_{n,\alpha}(k)$ hold, where $\nu(n) = \sup_{k \geq n} |\psi(k)|$, C and K are positive constants uniformly bounded on n .

At first we consider the case when the functions $p = p(x)$ and $s = s(x)$ on the period satisfy the inequality $s(x) \leq p(x)$. In our notation, the following assertion is true.

Theorem 1. *Let $(\psi; \beta) \in \Upsilon_{0,n}$ u $p, s \in \mathcal{P}^\gamma$, $s(x) \leq p(x)$, $x \in [0; 2\pi]$. Then, for all $n \in \mathbb{N}$ the inequality*

$$C_{p,s}\nu(n) \leq \mathcal{E}(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} \leq K_{p,s}\nu(n),$$

holds, where $C_{p,s}$ and $K_{p,s}$ are some constants that do not depend on n .

Proof. Suppose first that $p(x) \equiv s(x)$, $x \in [0; 2\pi]$. For an arbitrary function $f \in L_{\beta,p(\cdot)}^\psi$, the equality is true

$$\begin{aligned} f(x) - \hat{Z}_n(f; x) &= \sum_{k=1}^{n-1} \frac{\psi(n)}{\psi(k)} A_k(f; x) + \sum_{k=n}^{\infty} A_k(f; x) = \\ &= \sum_{k=1}^{\infty} \mu_{n,0} A_k(f_\beta^\psi; x) + \sum_{k=1}^{\infty} \tilde{\mu}_{n,0} \tilde{A}_k(f_\beta^\psi; x) := M_0(f_\beta^\psi) + \tilde{M}_0(\tilde{f}_\beta^\psi), \end{aligned} \quad (14)$$

where M_0 and \tilde{M}_0 are operators-multipliers, which are defined by the sequences (10) and (11) respectively, $\alpha = 0$.

According to the conditions of the theorem, the couples $(\psi; \beta)$ belong to the set $\Upsilon_{0,n}$, therefore, the sequence (10) and (11) satisfy the conditions of lemma A. Applying this lemma, given the inequalities (6), (12) and (13), for an arbitrary function $f \in L_{\beta,p(\cdot)}^\psi$ on the basis of the equality (14) we find

$$\begin{aligned} \|f(\cdot) - \hat{Z}_n(f; \cdot)\|_{p(\cdot)} &= \|M_0(f_\beta^\psi) + \tilde{M}_0(\tilde{f}_\beta^\psi)\|_{p(\cdot)} \leq \\ &\leq K\nu(n)(\|f_\beta^\psi\|_{p(\cdot)} + \|\tilde{f}_\beta^\psi\|_{p(\cdot)}) \leq C_p\nu(n), \end{aligned} \quad (15)$$

where C_p is positive constant that depends only on the function $p = p(x)$.

In the article [18] was shown that if $1 \leq s(x) \leq p(x) \leq \bar{p} < \infty$, then for an arbitrary function $f \in L^{p(\cdot)}$ the inequality hold

$$\|f\|_{s(\cdot)} \leq K_{s,p} \|f\|_{p(\cdot)}. \quad (16)$$

From the relations (15) and (16) we obtain the estimate

$$\mathcal{E}(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} \leq \mathcal{E}(L_{\beta,s(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} \leq C_{p,s} \nu(n), \quad (17)$$

where $C_{p,s}$ is positive constant that depends only on the functions $p = p(x)$ and $s = s(x)$.

We now obtain the lower estimate for the value of $\mathcal{E}(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)}$. If for given function $\psi(k)$ and the number $n \in \mathbb{N}$ there exist the natural number k_n , for which the equality

$$\nu(n) = \sup_{k \geq n} |\psi(k)| = \psi(k_n), \quad (18)$$

is true, then the corresponding lower estimate can be obtained with help of the function

$$f_n(x) = \frac{\psi(k_n)}{\|\cos k_n x\|_{p(\cdot)}} \cos(k_n x - \frac{\beta\pi}{2}). \quad (19)$$

Indeed, since

$$\|(f_n(x))_\beta\|_{p(\cdot)} = \left\| \frac{\cos k_n x}{\|\cos k_n x\|_{p(\cdot)}} \right\|_{p(\cdot)} = 1,$$

then the function $f_n \in L_{\beta,p(\cdot)}^\psi$ and

$$\begin{aligned} \mathcal{E}(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} &\geq \|f_n - \hat{Z}(f_n)\|_{s(\cdot)} = \\ &= \frac{\psi(k_n)}{\|\cos k_n x\|_{p(\cdot)}} \|\cos k_n x\|_{s(\cdot)} = C_{p,s} \nu(n). \end{aligned}$$

But if for the function $\psi(k)$ and the number $n \in \mathbb{N}$ there not exist a natural number k_n , for which the equality (18) holds, due to the limitation of the set $\{|\psi(k)|\}$ of values of the function $\psi(k)$ we will have

$$\nu(n) = \sup_{k \geq n} |\psi(k)| = \sup_{k \geq n} \{|\psi(k)|\}.$$

In this case, there exists a sequence k_j , $j \in \mathbb{N}$ such that $k_j \geq n$ and the numbers $\psi(k_j)$ don't decrease and converge to $\nu(n)$. Let $\mathbb{A} = \cup_j f_j(x)$, where the function $f_j(x)$ defined by the equality (19). Since $f_j \in L_{\beta,p(\cdot)}^\psi$ for any $j \in \mathbb{N}$, than

$$\mathcal{E}(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} = \sup_{f \in L_{\beta,p(\cdot)}^\psi} \|f - \hat{Z}_n(f)\|_{s(\cdot)} \geq \sup_{f \in \mathbb{A}} \|f - \hat{Z}_n(f)\|_{s(\cdot)} =$$

$$= \sup_{j \in \mathbb{N}} \frac{\psi(k_j)}{\|\cos k_j x\|_{p(\cdot)}} \|\cos k_j x\|_{s(\cdot)} = C_{p,s} \nu(n).$$

The theorem is proved.

We now obtain an estimate of the sequence $\mathcal{E}(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)}$ in the case where the function $p = p(x)$ and $s = s(x)$ on the period satisfy the inequality $p(x) < s(x)$. The following result gives the upper estimate.

Theorem 2. *Let $p, s \in \mathcal{P}^\gamma$, $p(x) < s(x)$, $x \in [0; 2\pi]$ and $(\psi; \beta) \in \Upsilon_{\alpha,n}$, $\alpha = 1/\underline{p} - 1/\bar{s}$. Then, for all $n \in \mathbb{N}$ the following inequality*

$$\mathcal{E}(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} \leq C_{p,s} n^\alpha \nu(n), \quad (20)$$

hold, where $C_{p,s}$ is a positive constant which independent of n .

Proof. For an arbitrary function $f \in L_{\beta,p(\cdot)}^\psi$, the equality

$$\begin{aligned} f(x) - \hat{Z}_n(f; x) &= \sum_{k=1}^{n-1} \frac{\psi(n)}{\psi(k)} A_k(f; x) + \sum_{k=n}^{\infty} A_k(f; x) = \\ &= \sum_{k=1}^{\infty} \mu_{n,\alpha} k^{-\alpha} A_k(f_\beta^\psi; x) + \sum_{k=1}^{\infty} \tilde{\mu}_{n,\alpha} k^{-\alpha} \tilde{A}_k(f_\beta^\psi; x) := M_\alpha(g_\alpha) + \tilde{M}_\alpha(\tilde{g}_\alpha), \end{aligned} \quad (21)$$

holds, where M_α and \tilde{M}_α are operators-multipliers, which are defined by the sequences (10) and (11) respectively, $\alpha = 1/\underline{p} - 1/\bar{s}$ and

$$\begin{aligned} g_\alpha(x) &:= \sum_{k=1}^{\infty} k^{-\alpha} A_k(f_\beta^\psi; x) = \frac{1}{\pi} \int_0^{2\pi} f_\beta^\psi(x+t) D_\alpha(t) dt, \\ \tilde{g}_\alpha(x) &:= \sum_{k=1}^{\infty} k^{-\alpha} \tilde{A}_k(f_\beta^\psi; x) = \frac{1}{\pi} \int_0^{2\pi} \tilde{f}_\beta^\psi(x+t) D_\alpha(t) dt, \end{aligned}$$

$D_\alpha(t)$ is function defined in Theorem B.

Since $f \in L_{\beta,p(\cdot)}^\psi$, then $f_\beta^\psi \in L^{p(\cdot)}$, and moreover $f_\beta^\psi \in L^{\underline{p}}$. By Theorem B we conclude that the convolution $g_\alpha(x)$ belongs $L^{\bar{s}}$, and moreover $g_\alpha \in L^{s(\cdot)}$. From the condition $(\psi; \beta) \in \Upsilon_{n,\alpha}$ by lemma A we conclude that the operator-multiplier M_α acts from $L^{s(\cdot)}$ to $L^{s(\cdot)}$ for any $s \in \mathcal{P}^\gamma$. Using analogous arguments for the function $\tilde{g}_\alpha(x)$, taking into account the inequalities (6), (9), (12) and (13), for an arbitrary function $f \in L_{\beta,p(\cdot)}^\psi$ on the basis of the equality (21) we find

$$\|f - \hat{Z}_n(f)\|_{s(\cdot)} \leq \|M_\alpha(g_\alpha)\|_{s(\cdot)} + \|\tilde{M}_\alpha(\tilde{g}_\alpha)\|_{s(\cdot)} \leq K n^\alpha \nu(n) (\|g_\alpha\|_{s(\cdot)} + \|\tilde{g}_\alpha\|_{s(\cdot)}) \leq$$

$$\leq C_{p,s} n^\alpha \nu(n) (\|f_\beta^\psi\|_{p(\cdot)} + \|\tilde{f}_\beta^\psi\|_{p(\cdot)}) \leq C_{p,s} n^\alpha \nu(n).$$

To make formulate the following assertion, which gives a lower estimate for the quantity $\mathcal{E}(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)}$ in the case, if the function $p = p(x)$ and $s = s(x)$ satisfy the inequality $p(x) < s(x)$ on the period, we need following definition.

Denoted by \mathfrak{B} the set of pairs $(\psi; \beta)$, such that for any $n \in \mathbb{N}$ the relations are true:

$$\sup_{n \leq k \leq 2n} \left| \frac{\nu(n)}{\psi(k)} \right| \leq C, \quad \sup_{m \in \mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\tau(k+1) - \tau(k)| \leq C,$$

where C is a positive constant which independent of n , $\nu(n) = \sup_{k \geq n} \psi(k)$ and

$$\tau(k) := \begin{cases} 0, & 1 \leq k \leq n-1, \\ \frac{\nu(n)}{\psi(k)}, & n \leq k \leq 2n. \end{cases} \quad (22)$$

Theorem 3. *Let $p, s \in \mathcal{P}^\gamma$, $p(x) < s(x)$, $x \in [0; 2\pi]$ and $(\psi; \beta) \in \mathfrak{B}$. Then, for all $n \in \mathbb{N}$ the following inequality is true*

$$\mathcal{E}_n(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} \geq C_{p,s} \nu(n) n^{1/\bar{p}-1/\underline{s}}, \quad (23)$$

where $C_{p,s}$ is a positive constant which independent of n .

Proof. For obtaining a lower estimate, let us show that for any positive integer n in class $L_{\beta,p(\cdot)}^\psi$ there exists a function f_n^* , for which the inequality is true

$$\|f_n^* - \hat{Z}_n\|_{s(\cdot)} \geq C_{p,g} \nu(n) n^{1/\bar{p}-1/\underline{s}}.$$

For this we fix $n \in \mathbb{N}$ and consider the function

$$f_n^*(x) = \sum_{k=n}^{2n} \psi(k) \cos(kx - \frac{\beta\pi}{2}).$$

Since

$$(f_n^*(x))_\beta^\psi = \sum_{k=n}^{2n} \cos kx = \frac{\sin nx/2 \cos(3n+1)x/2}{\sin x/2},$$

then using the relation (16) and also well-known inequality

$$\frac{x}{\pi} \leq \sin x/2, \quad \sin x \leq x, \quad x \in [0; \pi], \quad (24)$$

we obtain

$$\|(f_n^*)_\beta^\psi\|_{p(\cdot)} = \left\| \sum_{k=n}^{2n} \cos kx \right\|_{p(\cdot)} \leq K_p \left\| \sum_{k=n}^{2n} \cos kx \right\|_{\bar{p}} =$$

$$= \left(2 \int_0^\pi \left| \sum_{k=n}^{2n} \cos kx \right|^{\bar{p}} dx \right)^{1/\bar{p}} \leq \left(2 \int_0^\pi \left| \frac{\sin nx/2}{\sin x/2} \right|^{\bar{p}} dx \right)^{1/\bar{p}} \leq C_p^* n^{1-1/\bar{p}}.$$

This implies that the function

$$g_n^*(x) = \frac{n^{1/\bar{p}-1}}{C_p^*} f_n^*(x) = \frac{n^{1/\bar{p}-1}}{C_p^*} \sum_{k=n}^{2n} \psi(k) \cos(kx - \frac{\beta\pi}{2})$$

belongs to the class $L_{\beta,p(\cdot)}^\psi$.

Again using the inequalities (16) and (24), we find

$$\begin{aligned} \left\| \sum_{k=n}^{2n} \cos kx \right\|_{s(\cdot)} &\geq \left(2 \int_0^\pi \left| \frac{\sin nx/2 \cos(3n+1)x/2}{\sin x/2} \right|^{\underline{s}} dx \right)^{1/\underline{s}} \geq \\ &\geq C_s n^{1-1/\underline{s}} \left(\int_0^{\pi/2} (\cos x)^{\underline{s}} dx \right)^{1/\underline{s}} \geq K_s n^{1-1/\underline{s}}. \end{aligned} \quad (25)$$

If now by T_ψ we denote the operator-multiplier that generates a sequence (22), then by lemma A on condition $(\psi; \beta) \in \mathfrak{B}$ we will have

$$\begin{aligned} \left\| \sum_{k=n}^{2n} \cos(kx - \frac{\beta\pi}{2}) \right\|_{s(\cdot)} &= \left\| T_\psi \left(\sum_{k=n}^{2n} \frac{\psi(k)}{\nu(n)} \cos(kx - \frac{\beta\pi}{2}) \right) \right\|_{s(\cdot)} \leq \\ &\leq C \left\| \sum_{k=n}^{2n} \frac{\psi(k)}{\nu(n)} \cos(kx - \frac{\beta\pi}{2}) \right\|_{s(\cdot)}. \end{aligned}$$

Hence, considering the inequality (25) we find

$$\begin{aligned} \left\| \sum_{k=n}^{2n} \frac{\psi(k)}{\nu(n)} \cos(kx - \frac{\beta\pi}{2}) \right\|_{s(\cdot)} &\geq K \left\| \sum_{k=n}^{2n} \cos(kx - \frac{\beta\pi}{2}) \right\|_{s(\cdot)} \geq \\ &\geq C_s \left\| \sum_{k=n}^{2n} \cos kx \right\|_{\underline{s}} \geq K_s n^{1-1/\underline{s}}. \end{aligned} \quad (26)$$

Using the relation (26), we obtain

$$\begin{aligned} \mathcal{E}_n(L_{\beta,p(\cdot)}^\psi; \hat{Z}_n)_{s(\cdot)} &\geq \|g_n^* - \hat{Z}_n(g_n^*)\|_{s(\cdot)} = \left\| \frac{n^{1/\bar{p}-1}}{C_p^*} \sum_{k=n}^{2n} \psi(k) \cos(kx - \frac{\beta\pi}{2}) \right\|_{s(\cdot)} \geq \\ &\geq \frac{n^{1/\bar{p}-1}}{C_p^*} \nu(n) \left\| \sum_{k=n}^{2n} \frac{\psi(k)}{\nu(n)} \cos(kx - \frac{\beta\pi}{2}) \right\|_{s(\cdot)} \geq \\ &\geq C_{p,s} n^{1/\bar{p}-1} \nu(n) n^{1-1/\underline{s}} = C_{p,s} \nu(n) n^{1/\bar{p}-1/\underline{s}}. \end{aligned}$$

The theorem is proved.

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